

Periodic Wave Solutions of a Generalized KdV-mKdV Equation with Higher-Order Nonlinear Terms

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The Jacobin doubly periodic wave solution, the Weierstrass elliptic function solution, the bell-type solitary wave solution, the kink-type solitary wave solution, the algebraic solitary wave solution, and the triangular solution of a generalized Korteweg-de Vries-modified Korteweg-de Vries equation (GKdV-mKdV) with higher-order nonlinear terms are obtained by a generalized subsidiary ordinary differential equation method (Gsub-ODE method for short). The key ideas of the Gsub-ODE method are that the periodic wave solutions of a complicated nonlinear wave equation can be constructed by means of the solutions of some simple and solvable ODE which are called Gsub-ODE with higher-order nonlinear terms.

Key words: GKdV-mKdV Equation; Homogeneous Balance Method; New Auxiliary Equation Method; ODE with Higher-Degree Nonlinear Terms; Periodic Wave Solutions; Solitary Wave Solutions.

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1. Introduction

Many powerful methods to construct exact solutions of nonlinear evolution equations has been established and developed. Among these methods we mainly cite, for example, the inverse scattering transform method [1–2], the Bäcklund and Darboux transforms [3–9], the Hirota bilinear method [10–12], the variational iteration method [13–14], the collocation method [15–17], the Adomian Padé approximation [18], the Li group method [19–20], the tanh-function expansion method and its various extensions [21–25], the auxiliary ordinary differential equation method [26–28], the F-expansion method [29–30] and so on. For the nonlinear partial differential equation with higher-order nonlinear terms, many solitary wave solutions of them have been obtained by use of the above methods, however, periodic wave solutions of them have less been investigated.

In the present paper, we consider a generalized KdV-mKdV equation with higher-order nonlinear terms:

$$\frac{\partial u}{\partial t} + (\epsilon u^{-2P} + \delta u^{-P} + \alpha + \beta u^P + \gamma u^{2P}) \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \quad (1)$$

In fact, if one takes $\epsilon = \delta = 0$, (1) can be reduced to

$$\frac{\partial u}{\partial t} + (\alpha + \beta u^P + \gamma u^{2P}) \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \quad (2)$$

The aim of this article is to look for the travelling wave solutions of (1). When we make a transformation $u(x, t) = u(\xi)$, then (1) can be reduced to the following equation:

$$\frac{d^3 u}{d\xi^3} + (\epsilon u^{-2P} + \delta u^{-P} + \alpha + \beta u^P + \gamma u^{2P}) \frac{du}{d\xi} - c \frac{du}{d\xi} = 0, \quad (3)$$

where $\xi = x - ct$, and c is the wave speed.

These equations have potential applications to describe the propagation of atmospheric internal gravity waves. The ultimate source of turbulence in the atmosphere and oceans is a result of gravity waves which interact with critical layers [28, 31]. Making use of the Gsub-ODE method, many explicit exact solutions, which contain bell-profile solitary wave solutions, kink-profile solitary wave solutions, periodic solutions, and rational solutions, are obtained.

The rest of this paper is organized as follows. In Section 2, we describe the improved method. In Section 3,

we apply the improved method to the GKdV-mKdV equation and bring out many solutions. Conclusions will be presented in Section 4.

2. New Subsidiary Ordinary Differential Equation and its Solutions

The new ODE and its solutions are listed below, which can be examined directly. The Gsub-ODE is

$$\left(\frac{dF}{d\xi}\right)^2 = AF^{2-2P} + BF^{2-P} + CF^2 + DF^{2+2P} + EF^{2+2P}, \quad P > 0. \quad (4)$$

(4) admits of the following solutions:

(i) If $A = B = D = 0$, then (4) possesses a bell-shaped soliton solution

$$F_1 = \left(\sqrt{-\frac{C}{E}} \operatorname{sech}(\sqrt{CP}\xi) \right)^{\frac{1}{P}}, \quad C > 0, E < 0, \quad (5)$$

a triangular solution

$$F_2 = \left(\sqrt{-\frac{C}{E}} \sec(\sqrt{-CP}\xi) \right)^{\frac{1}{P}}, \quad C < 0, E > 0, \quad (6)$$

and a rational solution

$$F_3 = \left(\pm \frac{1}{\sqrt{EP}\xi} \right)^{\frac{1}{P}}, \quad C = 0, E > 0. \quad (7)$$

(ii) If $B = D = 0, A = \frac{C^2}{4E}$, then (4) possesses a kink-shaped soliton solution

$$F = \left(\pm \sqrt{-\frac{C}{E}} \tanh(\sqrt{CP}\xi) \right)^{\frac{1}{P}}, \quad C > 0, E < 0, \quad (8)$$

and a triangular solution

$$F = \left(\pm \sqrt{-\frac{C}{E}} \tan(\sqrt{-CP}\xi) \right)^{\frac{1}{P}}, \quad C < 0, E > 0. \quad (9)$$

(iii) If $B = D = 0$, then (4) admits three Jacobin elliptic function solutions

$$F = \left(\sqrt{-\frac{Cm^2}{E(2m^2-1)}} \operatorname{cn} \left(\sqrt{\frac{C}{2m^2-1}} P\xi \right) \right)^{\frac{1}{P}}, \quad (10)$$

$$C > 0, \quad A = \frac{C^2 m^2 (m^2 - 1)}{E(2m^2 - 1)^2},$$

$$F = \left(\sqrt{-\frac{C}{E(2-m^2)}} \operatorname{dn} \left(\sqrt{\frac{C}{2-m^2}} P\xi \right) \right)^{\frac{1}{P}}, \quad (11)$$

$$C > 0, \quad A = \frac{C^2(1-m^2)}{E(2-m^2)^2},$$

and

$$F = \left(\sqrt{-\frac{Cm^2}{E(m^2+1)}} \operatorname{sn} \left(\sqrt{\frac{-C}{m^2+1}} P\xi \right) \right)^{\frac{1}{P}}, \quad (12)$$

$$C < 0, \quad A = \frac{C^2 m^2}{E(m^2 + 1)^2}.$$

Remarks: When $m \rightarrow 1$, the Jacobin doubly periodic solutions (10) and (11) all degenerate to (5), and the Jacobin doubly periodic solution (12) degenerates to (8).

(iv) If $A = B = E = 0$, then (4) possesses a bell-shaped soliton solution

$$F = \left(-\frac{C}{D} \operatorname{sech}^2 \left(\frac{\sqrt{C}}{2} P\xi \right) \right)^{\frac{1}{P}}, \quad C > 0, D < 0, \quad (13)$$

a triangular solution

$$F = \left(-\frac{C}{D} \sec^2 \left(\frac{\sqrt{-C}}{2} P\xi \right) \right)^{\frac{1}{P}}, \quad C < 0, D > 0, \quad (14)$$

and a rational solution

$$F = \left(-\frac{1}{D(P\xi)^2} \right)^{\frac{1}{P}}, \quad C = 0, \quad D < 0. \quad (15)$$

(v) If $C = E = 0, D > 0$, then (4) admits a Weierstrass elliptic function solution

$$F = \left(\wp \left(\frac{\sqrt{D}}{2} P\xi, g_2, g_3 \right) \right)^{\frac{1}{P}}, \quad (16)$$

where $g_2 = -4\frac{B}{D}$ and $g_3 = -4\frac{A}{D}$ are called invariants of the Weierstrass elliptic function.

(vi) If $B = D = 0$, then (4) has some Weierstrass elliptic function solution

$$F = \left(\frac{1}{E} \wp(P\xi, g_2, g_3) - \frac{C}{3} \right)^{\frac{1}{2P}}, \quad (17)$$

where $g_2 = \frac{4C^2-12AE}{3}$, $g_3 = \frac{4C(-2C^2+9AE)}{27}$ are called invariants of the Weierstrass elliptic function,

$$F = \left(\frac{3A}{3\wp(P\xi, g_2, g_3) - C} \right)^{\frac{1}{p}}, \quad (18)$$

where $g_2 = \frac{4C^2-12AE}{3}$, $g_3 = \frac{4C(-2C^2+9AE)}{27}$ are called invariants of the Weierstrass elliptic function,

$$F = \left(\frac{\sqrt{12A\wp(P\xi, g_2, g_3) + 2A(2C + \pi)}}{12\wp(P\xi, g_2, g_3) + \pi} \right)^{\frac{1}{p}}, \quad (19)$$

where $g_2 = -\frac{1}{12}(5C\pi + 4C^2 + 33ACE)$, $g_3 = -\frac{4C}{216}(-21C^2\pi + 63AE\pi - 20C^3 + 27ACE)$, and $\pi = \frac{1}{2}(-5C \pm \sqrt{9C^2 - 36AE})$ are called invariants of the Weierstrass elliptic function,

$$F = \left(\frac{6\sqrt{A}\wp(P\xi, g_2, g_3) + C\sqrt{A}}{3\wp(P\xi, g_2, g_3)} \right)^{\frac{1}{p}}, \quad (20)$$

where $\wp(P\xi, g_2, g_3) = \frac{d\wp(P\xi, g_2, g_3)}{d\xi}$, $g_2 = \frac{C^2}{12} + AE$, $g_3 = \frac{C(36AEC^2)}{216}$ are called invariants of the Weierstrass elliptic function,

$$F = \left(\frac{3\sqrt{E^{-1}}\wp(P\xi, g_2, g_3)}{6\wp(P\xi, g_2, g_3) + C} \right)^{\frac{1}{p}}, \quad (21)$$

where $\wp(P\xi, g_2, g_3) = \frac{d\wp(P\xi, g_2, g_3)}{d\xi}$, $g_2 = \frac{C^2}{12} + AE$, and $g_3 = \frac{AEC^3}{6}$ are called invariants of the Weierstrass elliptic function.

(vii) If $B = D = 0$, $A = \frac{5C^2}{36E}$, then (4) has a Weierstrass elliptic function solution

$$F = \left(\frac{C\sqrt{-\frac{15C}{2E}}\wp(P\xi, g_2, g_3)}{3\wp(P\xi, g_2, g_3) + C} \right)^{\frac{1}{p}}, \quad (22)$$

where $g_2 = \frac{2C^2}{9}$ and $g_3 = \frac{C^3}{54}$ are called invariants of the Weierstrass elliptic function.

(viii) If $A = B = 0$, then (4) admits of three positive solution

$$F = \left(\frac{1}{\cosh P\sqrt{C}\xi - \frac{D}{2C}} \right)^{\frac{1}{p}}, \quad (23)$$

where $C > 0$, $D < 2C$ and $E = \frac{D^2}{4C} - C$,

$$F = \left(\sqrt{\frac{C}{E}} \left(\frac{1}{2} \pm \frac{1}{2} \tanh \frac{1}{2} P\sqrt{C}\xi \right) \right)^{\frac{1}{p}}, \quad (24)$$

where $C > 0$, $E > 0$, $D = -2\sqrt{CE}$,

$$F = \left(\frac{1}{(\frac{1}{2}P\xi)^2 - E} \right)^{\frac{1}{p}}, \quad (25)$$

where $C = 0$, $D = 1$, $E < 0$.

(ix) If $E < 0$, $C > 0$, $D < 0$, $A = B = 0$, and $D^2 - 4CE > 0$, then (4) has a bell-profile solution

$$F = \left\{ 2C \operatorname{sech}^2 \sqrt{C}\xi \left[2\sqrt{D^2 - 4CE} + (\sqrt{D^2 - 4CE} + D) \operatorname{sech}^2 \sqrt{C}\xi \right]^{-1} \right\}^{\frac{1}{p}} \quad (26)$$

and a singular solution

$$F = \left\{ 2C \operatorname{csch}^2 (\pm \sqrt{C}\xi) \left[2\sqrt{D^2 - 4CE} + (\sqrt{D^2 - 4CE} - D) \operatorname{csch}^2 (\pm \sqrt{C}\xi) \right]^{-1} \right\}^{\frac{1}{p}}. \quad (27)$$

(x) If $E < 0$, $C < 0$, $D > 0$, $A = B = 0$, $D^2 - 4CE > 0$, then (4) has a triangular periodic solution

$$F = \left\{ -2C \sec^2 \sqrt{-C}\xi \left[2\sqrt{D^2 - 4CE} - (\sqrt{D^2 - 4CE} - D) \sec^2 \sqrt{-C}\xi \right]^{-1} \right\}^{\frac{1}{p}} \quad (28)$$

and a singular triangular periodic solution

$$F = \left\{ 2C \csc^2 (\pm \sqrt{-C}\xi) \left[2\sqrt{D^2 - 4CE} - (\sqrt{D^2 - 4CE} + D) \csc^2 [\pm \sqrt{-C}\xi] \right]^{-1} \right\}^{\frac{1}{p}}. \quad (29)$$

(xi) If $C < 0$, $D > 0$, $B = 0$, $A = \frac{8C^2}{27D}$ and $E = \frac{D^2}{4C}$, then (4) has a kink-profile solution

$$F = \left(-\frac{8C \tanh^2 \left[\pm \sqrt{-\frac{C}{3}}\xi \right]}{3D(3 + \tanh^2 \left[\pm \sqrt{-\frac{C}{3}}\xi \right])} \right)^{\frac{1}{p}} \quad (30)$$

and a singular solution

$$F = \left(-\frac{8C \coth^2 \left[\pm \sqrt{-\frac{C}{3}}\xi \right]}{3D(3 + \coth^2 \left[\pm \sqrt{-\frac{C}{3}}\xi \right])} \right)^{\frac{1}{p}}. \quad (31)$$

(xii) If $C > 0$, $D < 0$, $B = 0$, $A = \frac{C^2}{27D}$ and $E = \frac{D^2}{4C}$, then (4) has a triangular periodic solution

$$F = \left(\frac{8C \tan^2 \left[\pm \sqrt{\frac{C}{3}} \xi \right]}{3D(3 - \tan^2 \left[\pm \sqrt{\frac{C}{3}} \xi \right])} \right)^{\frac{1}{P}} \quad (32)$$

and a singular solution

$$F = \left(\frac{8C \cot^2 \left[\pm \sqrt{\frac{C}{3}} \xi \right]}{3D(3 - \cot^2 \left[\pm \sqrt{\frac{C}{3}} \xi \right])} \right)^{\frac{1}{P}}. \quad (33)$$

3. Exact Solutions of GKdV-mKdV Equation with Higher-Order Nonlinear Terms

We suppose that $u(\xi)$ can be expressed by

$$u(\xi) = \mu F^m(\xi), \quad \mu > 0, \quad (34)$$

where $F(\xi)$ satisfies (4). Considering the homogeneous balance between $u^{\pm 2P} \frac{du}{d\xi}$ and $\frac{d^3 u}{d\xi^3}$ in (3) ($\pm 2Pm + m \pm P = m \pm 3P \Rightarrow m = 1$), we can simply suppose that

$$u(\xi) = \mu F(\xi), \quad \mu > 0. \quad (35)$$

Using (35) and (4), we derive that

$$\begin{aligned} \frac{d^3 u}{d\xi^3} &= \mu \frac{d^3 F}{d\xi^3} = \frac{1}{2} \mu \left[(2A - 6AP + 4AP^2)F^{-2P} \right. \\ &+ (2B - 3BP + BP^2)F^{-P} + 2C + (2D + 3DP + DP^2)F^P \\ &\left. + (2E + 6EP + 4EP^2)F^{2P} \right] \frac{dF}{d\xi}. \end{aligned} \quad (36)$$

With the help of Maple 10, by substituting (36) and (35) into (3), and setting the coefficients of F^{-2P} , F^{-P} , F^0 , F^P , F^{2P} to zero, we obtain a system of algebraic equations:

$$\begin{aligned} F^{-2P}: & \quad A + 2AP^2 - 3AP + \varepsilon \mu^{(-2P)} = 0, \\ F^{-P}: & \quad -\frac{3BP}{2} + B + \delta \mu^{(-P)} + \frac{BP^2}{2} = 0, \\ F^0: & \quad \alpha + C - c = 0, \\ F^P: & \quad \frac{3DP}{2} + D + \beta \mu^P + \frac{DP^2}{2} = 0, \\ F^{2P}: & \quad 3EP + 2EP^2 + \gamma \mu^{(2P)} + E = 0. \end{aligned} \quad (37)$$

Solving this set of algebraic equations above, and we find the following solution:

$$\begin{aligned} A &= -\frac{\varepsilon \mu^{(-2P)}}{1 + 2P^2 - 3P}, \quad B = -\frac{2\delta \mu^{(-P)}}{P^2 - 3P + 2}, \\ C &= c - \alpha, \quad D = -\frac{2\beta \mu^P}{3P + 2 + P^2}, \\ E &= -\frac{\gamma \mu^{(2P)}}{1 + 3P + 2P^2}, \end{aligned} \quad (38)$$

with $P > 1$, and $A = A$, $B = B$, $\varepsilon = \delta = 0$, $C = c - \alpha$, $D = -\frac{1}{3}\beta \mu$ and $E = -\frac{1}{6}\gamma \mu^2$ with $P = 1$.

Considering the six parameters A , B , C , D , and E , we present some types of solutions in the following cases.

(i) If $A = B = D = 0$, we have $\varepsilon = \delta = \beta = 0$, then (3) is reduced to

$$\frac{d^3 u}{d\xi^3} + (\alpha + \gamma u^{2P}) \frac{du}{d\xi} - c \frac{du}{d\xi} = 0, \quad (39)$$

which admits a bell-shaped soliton solution

$$u_1 = \left[\sqrt{\frac{(c - \alpha)(1 + 3P + 2P^2)}{\gamma}} \cdot \operatorname{sech}(\sqrt{c - \alpha} P(x - ct)) \right]^{\frac{1}{P}}, \quad \alpha < c, \quad \gamma > 0, \quad (40)$$

a triangular solution

$$u_2 = \left[\sqrt{\frac{(c - \alpha)(1 + 3P + 2P^2)}{\gamma}} \cdot \sec(\sqrt{\alpha - c} P(x - ct)) \right]^{\frac{1}{P}}, \quad \alpha > c, \quad \gamma < 0, \quad (41)$$

and a rational solution

$$u_3 = \left(\pm \sqrt{\frac{1 + 3P + 2P^2}{\sqrt{-\gamma} P(x - ct)}} \right)^{\frac{1}{P}}, \quad \alpha = c, \quad \gamma < 0. \quad (42)$$

Figure 1 shows a spatial structure of the bell-shaped soliton solution u_1 .

(ii) If $B = D = 0$, $A = \frac{C^2}{4E}$, we have $\delta = \beta = 0$, then (3) becomes

$$\frac{d^3 u}{d\xi^3} + (\varepsilon \mu^{-2P} + \alpha + \gamma \mu^{2P}) \frac{du}{d\xi} - c \frac{du}{d\xi} = 0, \quad (43)$$

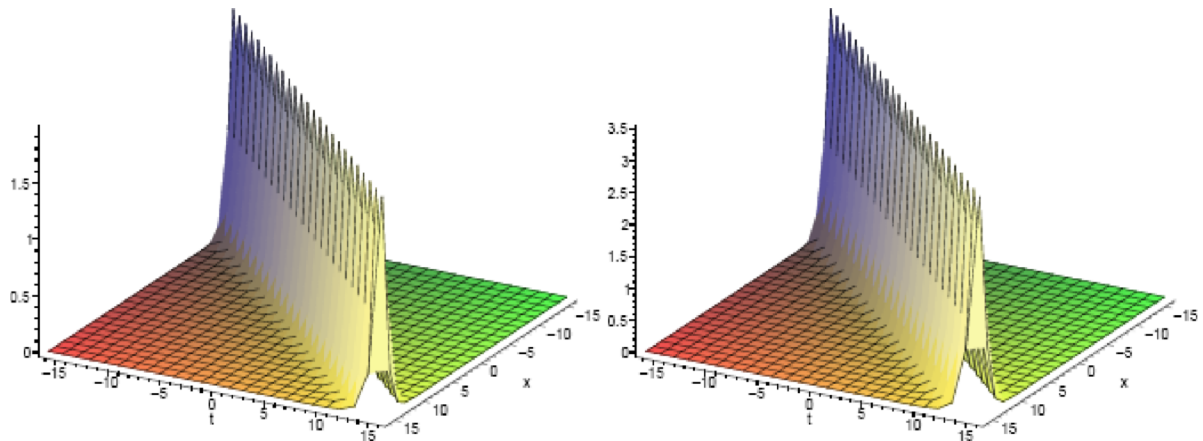


Fig. 1 (colour online). Structure of soliton solution u_1 (left) and $(u_1)^P$ (right), with parameters $\varepsilon = 1$, $c = 2$, $\alpha = -3$, $\gamma = 1$, and $P = 3$.

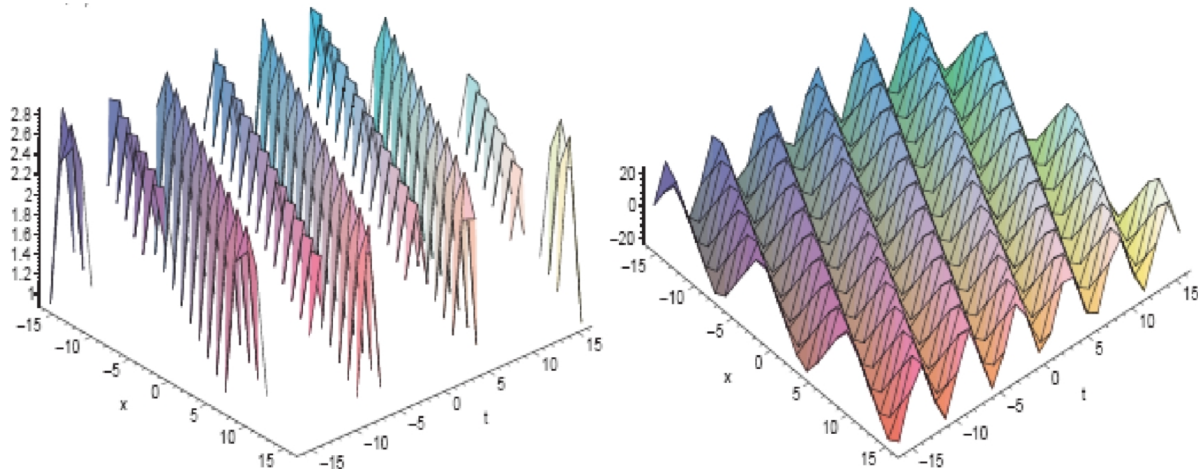


Fig. 2 (colour online). Structure of Jacobin periodic solution u_6 (left) and $(u_6)^P$ (right), with parameters $m = 0.8$, $c = -2$, $\alpha = 3$, $\gamma = -1$, and $P = 3$.

which has a kink-shaped soliton solution

$$u_4 = \left[\pm \sqrt{\frac{(c - \alpha)(1 + 3P + 2P^2)}{\gamma}} \cdot \tanh(\sqrt{c - \alpha}P(x - ct)) \right]^{\frac{1}{P}}, \quad c > \alpha, \quad \gamma > 0, \quad (44)$$

and a triangular solution

$$u_5 = \left[\pm \sqrt{\frac{(c - \alpha)(1 + 3P + 2P^2)}{\gamma}} \cdot \tan(\sqrt{\alpha - c}P(x - ct)) \right]^{\frac{1}{P}}, \quad c < \alpha, \quad \gamma < 0, \quad (45)$$

where $c = \alpha \pm \sqrt{\frac{4\varepsilon\gamma}{(1+2P^2+3P)(1+2P^2-3P)}}$, when $P > 1$;

$c = \alpha \pm \mu\sqrt{-\frac{2A\gamma}{3}}$, when $P = 1$.

(iii) If $B = D = 0$, we have $\delta = \beta = 0$, then (43) admits three Jacobin elliptic function solutions

$$u_6 = \left[\sqrt{\frac{(c - \alpha)(1 + 3P + 2P^2)m^2}{\gamma(2m^2 - 1)}} \cdot \text{cn}\left(\sqrt{\frac{c - \alpha}{2m^2 - 1}}P(x - ct)\right) \right]^{\frac{1}{P}}, \quad (46)$$

where $c = \alpha + \sqrt{\frac{\varepsilon\gamma(2m^2 - 1)^2}{(1+2P^2+3P)(1+2P^2-3P)m^2(m^2 - 1)}}$, when $P > 1$; $c = \alpha + \sqrt{\frac{A\gamma\mu^2(2m^2 - 1)^2}{6m^2(1 - m^2)}}$, when $P = 1$. Figure 2 shows a spatial structure of the Jacobin elliptic func-

tion solution u_6 .

$$u_7 = \left[\sqrt{\frac{(c-\alpha)(1+3P+2P^2)}{\gamma(2-m^2)}} \cdot \operatorname{dn} \left(\sqrt{\frac{c-\alpha}{2-m^2}} P(x-ct) \right) \right]^{\frac{1}{P}}, \quad (47)$$

where $c = \alpha + \sqrt{\frac{\varepsilon\gamma(2-m^2)^2}{(1+2P^2+3P)(1+2P^2-3P)(1-m^2)}}$, when $P > 1$; $c = \alpha + \sqrt{\frac{A\gamma\mu^2(2-m^2)^2}{6(m^2-1)}}$, when $P = 1$.

$$u_8 = \left[\sqrt{\frac{(c-\alpha)(1+3P+2P^2)m^2}{\gamma(m^2+1)}} \cdot \operatorname{sn} \left(\sqrt{\frac{\alpha-c}{m^2+1}} P(x-ct) \right) \right]^{\frac{1}{P}}, \quad (48)$$

where $c = \alpha - \sqrt{\frac{\varepsilon\gamma(m^2+1)^2}{(1+2P^2+3P)(1+2P^2-3P)(1-m^2)}}$, when $P > 1$; $c = \alpha - \sqrt{-\frac{A\gamma\mu^2(m^2+1)^2}{6m^2}}$, when $P = 1$.

Remarks: When $m \rightarrow 1$, the Jacobin doubly periodic solutions (46) and (47) all degenerate to (40), and the Jacobin doubly periodic solution (48) degenerates to (44).

(iv) If $A = B = E = 0$, we have $\varepsilon = \delta = \gamma = 0$, then (3) is reduced to

$$\frac{d^3 u}{d\xi^3} + (\alpha + \beta u^P) \frac{du}{d\xi} - c \frac{du}{d\xi} = 0, \quad (49)$$

which admits a bell-shaped soliton solution

$$u_9 = \left[\frac{(c-\alpha)(P^2+3P+2)}{2\beta} \cdot \operatorname{sech}^2 \left(\frac{\sqrt{c-\alpha}}{2} P(x-ct) \right) \right]^{\frac{1}{P}}, \quad c > \alpha, \beta > 0, \quad (50)$$

a triangular solution

$$u_{10} = \left[\frac{(c-\alpha)(P^2+3P+2)}{2\beta} \cdot \sec^2 \left(\frac{\sqrt{\alpha-c}}{2} P(x-ct) \right) \right]^{\frac{1}{P}}, \quad c < \alpha, \beta > 0, \quad (51)$$

and a rational solution

$$u_{11} = \left(\frac{P^2+3P+2}{2\beta(P(x-ct))^2} \right)^{\frac{1}{P}}, \quad c = \alpha, \beta > 0. \quad (52)$$

(v) If $B = D = 0$, we have $\delta = \beta = 0$, then (3) is reduced to (43) which has some Weierstrass elliptic function solution

$$u_{12} = \left[\frac{\alpha-c}{3} - \frac{1+3P+2P^2}{\gamma} \wp(P\xi, g_2, g_3) \right]^{\frac{1}{2P}}, \quad (53)$$

$$u_{13} = \left[\frac{3\varepsilon(1+2P^2-3P)^{-1}}{\alpha-c+3\wp(P\xi, g_2, g_3)} \right]^{\frac{1}{2P}}, \quad P > 1, \quad (54)$$

where $g_2 = \frac{4(c-\alpha)^2}{3} + \frac{4\varepsilon\gamma}{(1+2P^2+3P)(1+2P^2-3P)}$ and $g_3 = \frac{8(\alpha-c)^3}{27} + \frac{4\varepsilon\gamma(\alpha-c)}{3(1+2P^2+3P)(1+2P^2-3P)}$, when $P > 1$; $g_2 = \frac{4(c-\alpha)^2}{3} - \frac{4A\gamma\mu^2}{(1+2P^2+3P)}$ and $g_3 = \frac{8(\alpha-c)^3}{27} - \frac{4A\gamma\mu^2(\alpha-c)}{3(1+2P^2+3P)}$, when $P = 1$. Figure 3 shows a spatial structure of the Weierstrass elliptic function solution u_{12} .

$$u_{14} = \left[\frac{\sqrt{12\varepsilon\wp(P\xi, g_2, g_3) + 2\varepsilon(2c-2\alpha+\pi)}}{(1+2P^2-3P)(12\wp(P\xi, g_2, g_3) + \pi)} \right]^{\frac{1}{P}}, \quad P > 1, \quad (55)$$

where

$\pi = \frac{5}{2}(\alpha-c) \pm \sqrt{\frac{9}{4}(c-\alpha)^2 - \frac{6\varepsilon\gamma}{(1+2P^2-3P)(1+2P^2+3P)}}$, $g_2 = \frac{c-\alpha}{12} \left(\frac{33\varepsilon\gamma}{(1+2P^2+3P)(1+2P^2-3P)} - 5\pi + 4(\alpha-c) \right)$ and $g_3 = \frac{4(c-\alpha)}{216} \left(21(c-\alpha)^2\pi + \frac{63\varepsilon\gamma\pi + 27\varepsilon\gamma(c-\alpha)}{(1+2P^2-3P)(1+2P^2+3P)} + 20(c-\alpha)^3 \right)$ are called invariants of the Weierstrass elliptic function.

$$u_{15} = \left[\frac{\varepsilon\gamma(6\wp(P\xi, g_2, g_3) + c - \alpha)}{3(1+2P^2-3P)\wp'(P\xi, g_2, g_3)} \right]^{\frac{1}{P}}, \quad P > 1, \quad (56)$$

$$u_{16} = \left[\frac{3\sqrt{-\frac{1+2P^2+3P}{\gamma}}\wp'(P\xi, g_2, g_3)}{6\wp(P\xi, g_2, g_3) + c - \alpha} \right]^{\frac{1}{P}}, \quad (57)$$

where $\wp'(P\xi, g_2, g_3) = \frac{d\wp(P\xi, g_2, g_3)}{d\xi}$,

$g_2 = \frac{(c-\alpha)^2}{12} - \frac{\varepsilon\gamma}{(1+2P^2-3P)(1+2P^2+3P)}$, and $g_3 = \frac{\alpha-c}{216} \left(\frac{36\varepsilon\gamma}{(1+2P^2-3P)(1+2P^2+3P)} + (c-\alpha)^2 \right)$, when $P > 1$; $g_2 = \frac{(c-\alpha)^2}{12} + \frac{A\mu^2\gamma}{6}$ and $g_3 = \frac{c-\alpha}{36} (A\mu^2\gamma + (c-\alpha)^2)$, when $P = 1$.

(vi) If $B = D = 0$, $A = \frac{5C^2}{36E}$, we have $\delta = \beta = 0$, then (3) becomes (43), which has a Weierstrass elliptic

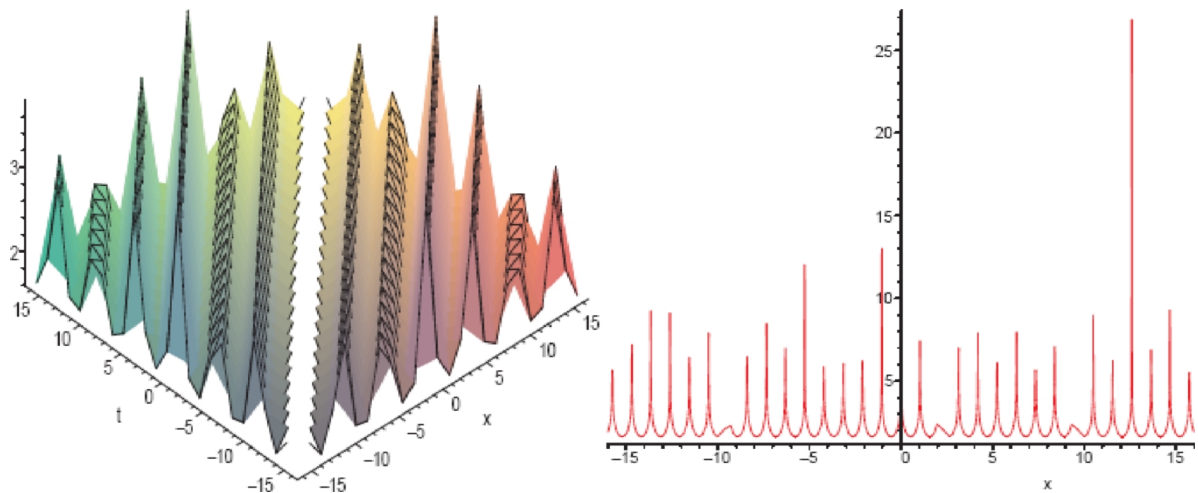


Fig. 3 (colour online). Structure of Weierstrass periodic solution u_{12} (left) and its initial value (right), with parameters $\varepsilon = 1$, $c = 1$, $\alpha = 2$, $\gamma = -1$, and $P = 3$.

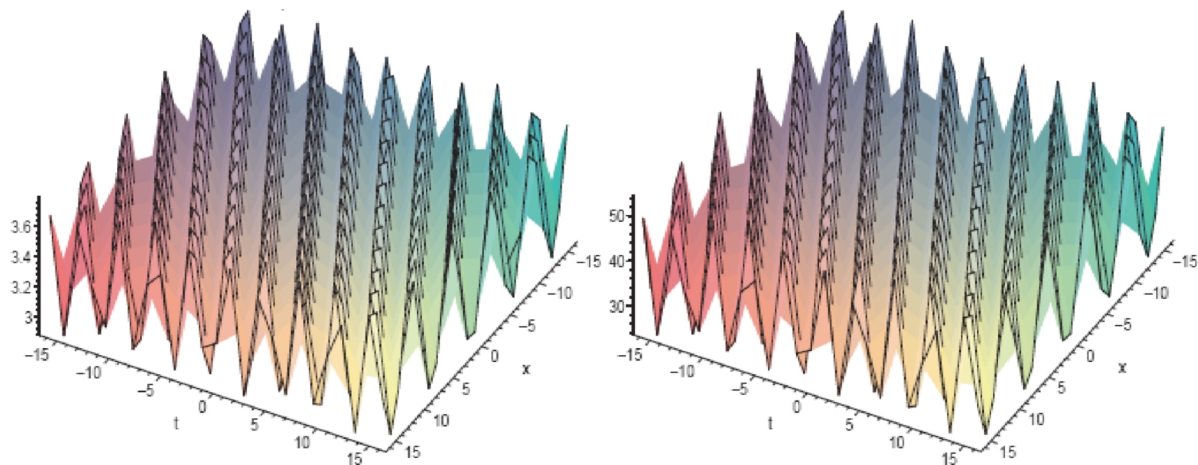


Fig. 4 (colour online). Structure of Weierstrass periodic solution u_{17} (left) and $(u_{17})^P$ (right), with parameters $\varepsilon = 1$, $c = 2$, $\alpha = -3$, $\gamma = 1$, and $P = 3$.

function solution

$$u_{17} = \left[\sqrt{\frac{15(c-\alpha)(1+2P^2+3P)}{2\gamma}} \cdot \frac{(c-\alpha)\wp(P\xi, g_2, g_3)}{3\wp(P\xi, g_2, g_3) + c - \alpha} \right]^{\frac{1}{P}}, \quad (58)$$

where $c = \alpha \pm \sqrt{\frac{-36\varepsilon\gamma}{5(1+2P^2+3P)(1+2P^2-3P)}}$, when $P > 1$;
 $c = \alpha \pm \sqrt{\frac{6A\mu^2\gamma}{5}}$, when $P = 1$. $g_2 = \frac{2(c-\alpha)^2}{9}$ and $g_3 =$

$\frac{(c-\alpha)^3}{54}$ are called invariants of the Weierstrass elliptic function. Figure 4 shows a spatial structure of the Weierstrass elliptic function solution u_{17} .

(vii) If $A = B = 0$, we have $\varepsilon = \delta = 0$, then (3) becomes

$$\frac{d^3u}{d\xi^3} + (\alpha + \beta u^P + \gamma u^{2P}) \frac{du}{d\xi} - c \frac{du}{d\xi} = 0, \quad (59)$$

which has a bell-profile solution

$$u_{18} = \left(\frac{2(c-\alpha) \operatorname{sech}^2 \sqrt{c-\alpha} \xi}{2\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} + \left(\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} + \frac{2\beta}{3P+2+P^2}\right) \operatorname{sech}^2 \sqrt{c-\alpha} \xi} \right)^{\frac{1}{P}} \quad (60)$$

and a singular solution

$$u_{19} = \left(\frac{2(c - \alpha) \operatorname{csch}^2(\pm \sqrt{c - \alpha} \xi)}{2\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} + \left(\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} + \frac{2\beta}{3P+2+P^2}\right) \operatorname{csch}^2(\pm \sqrt{c - \alpha} \xi)} \right)^{\frac{1}{p}}, \quad (61)$$

where $\gamma > 0$, $\beta > 0$, and $0 < c - \alpha < \frac{\beta^2(2+3P+P^2)^2}{\gamma(1+3P+2P^2)}$ in (60) and (61).

A triangular periodic solution

$$u_{20} = \left(\frac{2(\alpha - c) \sec^2 \sqrt{\alpha - c} \xi}{2\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} - \left(\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} - \frac{2\beta}{3P+2+P^2}\right) \sec^2 \sqrt{\alpha - c} \xi} \right)^{\frac{1}{p}} \quad (62)$$

and a singular triangular periodic solution

$$u_{21} = \left(\frac{2(c - \alpha) \csc^2(\pm \sqrt{\alpha - c} \xi)}{2\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} - \left(\sqrt{\left(\frac{2\beta}{3P+2+P^2}\right)^2 + \frac{4(c-\alpha)\gamma}{1+3P+2P^2}} - \frac{2\beta}{3P+2+P^2}\right) \csc^2(\pm \sqrt{\alpha - c} \xi)} \right)^{\frac{1}{p}}, \quad (63)$$

where $\gamma > 0$, $\beta < 0$, and $0 < c - \alpha < \frac{\beta^2(2+3P+P^2)^2}{\gamma(1+3P+2P^2)}$ in (62) and (63).

(viii) If $A = 0$, $B = \frac{8C^2}{27D}$, and $E = \frac{D^2}{4C}$, we have $\varepsilon = 0$, $\delta = \frac{2\beta^3(P^2+3P+2)^5(3P-P^2-2)}{27\gamma^2(1+3P+2P^2)}$. Then (3) becomes

$$\frac{d^3 u}{d\xi^3} + (\delta u^{-P} + \alpha + \beta u^P + \gamma u^{2P}) \frac{du}{d\xi} - c \frac{du}{d\xi} = 0, \quad (64)$$

which has a kink-profile solution

$$u_{22} = \left(\frac{4(2 + P^2 + 3P)(c - \alpha) \tanh^2 \left[\pm \sqrt{\frac{\alpha - c}{3}} \xi \right]}{3\beta \left(3 + \tanh^2 \left[\pm \sqrt{\frac{\alpha - c}{3}} \xi \right] \right)} \right)^{\frac{1}{p}}, \quad (65)$$

and singular solution

$$u_{23} = \left(\frac{4(2 + P^2 + 3P)(c - \alpha) \coth^2 \left[\pm \sqrt{\frac{\alpha - c}{3}} \xi \right]}{3\beta \left(3 + \coth^2 \left[\pm \sqrt{\frac{\alpha - c}{3}} \xi \right] \right)} \right)^{\frac{1}{p}}, \quad (66)$$

where $c < \alpha$, $\beta < 0$ in (65) and (66).

A triangular periodic solution

$$u_{24} = \left(\frac{4(2 + P^2 + 3P)(c - \alpha) \tan^2 \left[\pm \sqrt{\frac{c - \alpha}{3}} \xi \right]}{3\beta \left(3 - \tan^2 \left[\pm \sqrt{\frac{c - \alpha}{3}} \xi \right] \right)} \right)^{\frac{1}{p}}, \quad (67)$$

and a singular solution

$$u_{25} = \left(\frac{4(2 + P^2 + 3P)(c - \alpha) \cot^2 \left[\pm \sqrt{\frac{c - \alpha}{3}} \xi \right]}{3\beta \left(3 - \cot^2 \left[\pm \sqrt{\frac{c - \alpha}{3}} \xi \right] \right)} \right)^{\frac{1}{p}}, \quad (68)$$

where $c > \alpha$, $\beta > 0$ in (67) and (68).

4. Summary and Discussions

With the aid of symbolic computation, a generalized auxiliary equation method with higher-degree nonlinear terms is proposed to construct periodic wave solutions and solitary wave solutions of a GKdV-mKdV

equation with higher-order nonlinear terms. The types of solution depend on the values of ε , δ , α , β , and γ , which are the coefficients of the GKdV-mKdV with higher-order nonlinear terms. Most importantly, we can successfully obtain in a unified way many periodic wave solutions of the GKdV-mKdV equation with higher-order nonlinear terms. This method can also be extended to seek more solitary wave solutions, periodic

wave solutions, and other formal solutions of given nonlinear partial differential equations with any-order nonlinear terms.

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